15.6. Triple integrals : examples

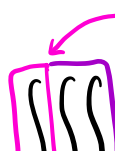Note There are two ways to set up a triple integral.

(1) Projection method.

Step 1. Choose the innermost variable.

Step 2. Find the bounds for the outer double integral by looking at the shadow

Step 3. Find the bounds for the innermost integral.

(2) Cross section method

Step 1. Choose the outermost variable.

Step 2. Find the bounds for the outermost integral by looking at extreme values.

Step 3. Find the bounds for the inner double integral by looking at the cross section.
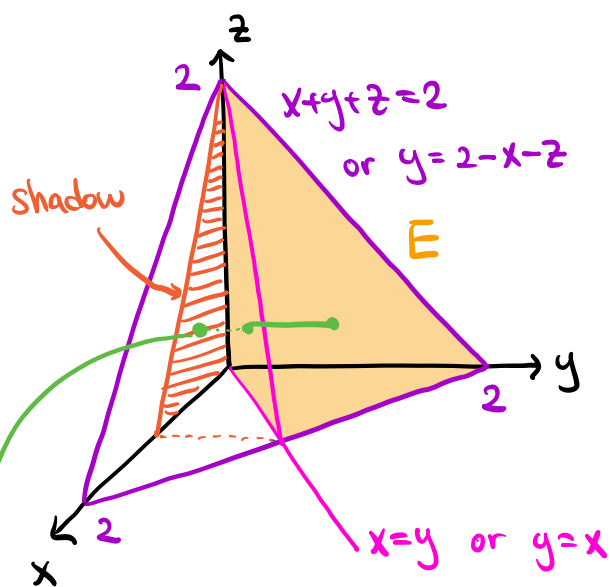
Remark The projection method requires a 3-dimensional sketch, but with minimum algebra.

The cross section method only requires a 2-dimensional sketch, but with heavy algebra.

For the exam, you can use the method of your preference.

Ex Let E be the solid bounded by the planes

$x=0$, $z=0$, $x=y$, $x+y+z=2$. Express the volume
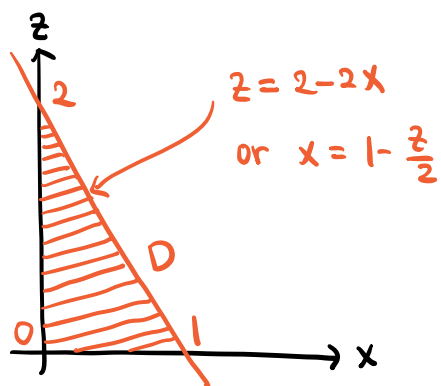
of E as a triple integral in the order $dy\,dx\,dz$.

Sol 1 (Projection method)

$x+y+z=2$: a plane

$\begin{cases} x\text{-intercept} = 2 \\ y\text{-intercept} = 2 \\ z\text{-intercept} = 2. \end{cases}$

For the outer double integral, look at the shadow D
on the xz-plane:

The bounds are given by

$0 \le z \le 2$, $0 \le x \le 1-\frac{z}{2}$.

For each point on D: $x \le y \le 2-x-z$.

$$\Rightarrow \text{Vol}(E) = \iiint_E 1\,dV = \boxed{\int_0^2 \int_0^{1-\frac{z}{2}} \int_x^{2-x-z} 1\,dy\,dx\,dz.}$$

Sol 2 (Cross section method)

The outermost integral is with respect to dz.
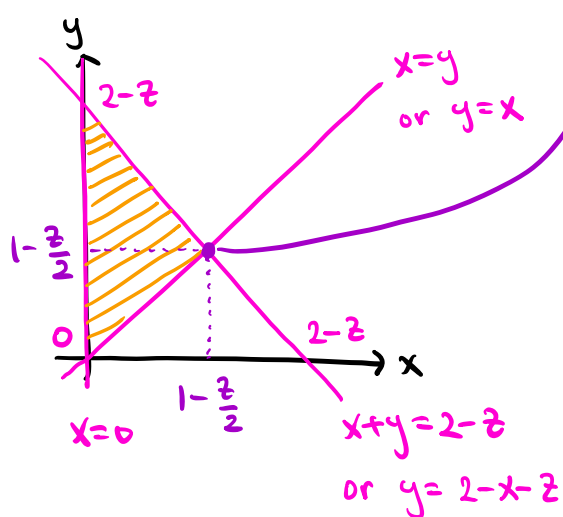
The boundary equations: $x=0$, $z=0$, $x=y$, $x+y+z=2$.

The extreme values occur at the intersections:

$x=0$, $x=y$, $x+y+z=2 \Rightarrow z=2$.

The bounds for z are $0 \le z \le 2$.

For the inner double integral, look at the cross

section with <u>constant</u> z.

The boundary equations: $x=0$, $x=y$, $x+y=2-z$.

Intersection: $x=y$ and $x+y=2-z$.
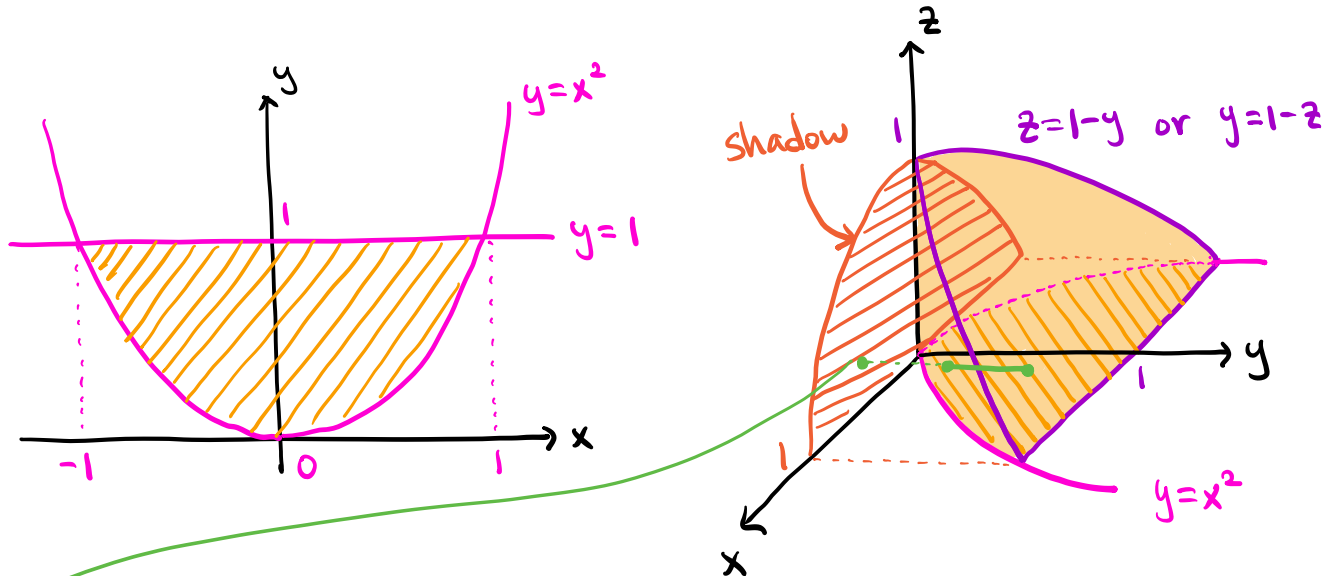
$\Rightarrow x=y=1-\dfrac{z}{2}$.

The bounds are given by

$0 \le x \le 1-\dfrac{z}{2}$, $x \le y \le 2-x-z$.

$$\Rightarrow \text{Vol}(E) = \iiint_E 1\, dV = \boxed{\int_0^2 \int_0^{1-\frac{z}{2}} \int_x^{2-x-z} 1\, dy\, dx\, dz}$$
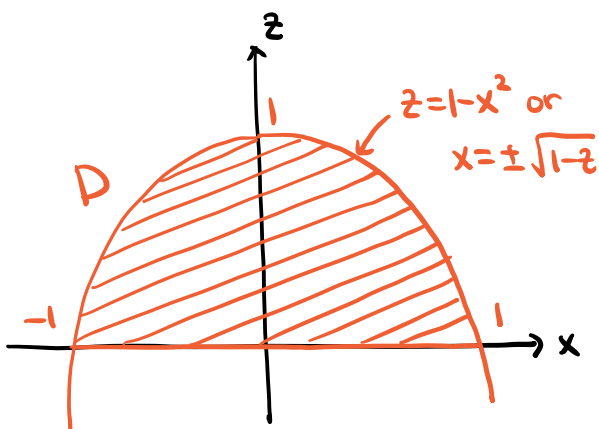
Ex Rewrite the integral $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} 1\,dz\,dy\,dx$ in the order $dy\,dx\,dz$.

Sol 1 (Projection method)

The bounds are $-1 \leq x \leq 1$, $x^2 \leq y \leq 1$, $0 \leq z \leq 1-y$.

For the outer double integral, look at the shadow D on the xz-plane:

The curve is given by $y = x^2$ and $z = 1-y$

$\Rightarrow z = 1-x^2$ or $x = \pm\sqrt{1-z}$

The bounds are given by

$0 \leq z \leq 1$, $-\sqrt{1-z} \leq x \leq \sqrt{1-z}$

For each point on D: $x^2 \leq y \leq 1-z$.

$\Rightarrow \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} 1\,dz\,dy\,dx = \boxed{\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} 1\,dy\,dx\,dz}$

Sol 2 (Cross section method)

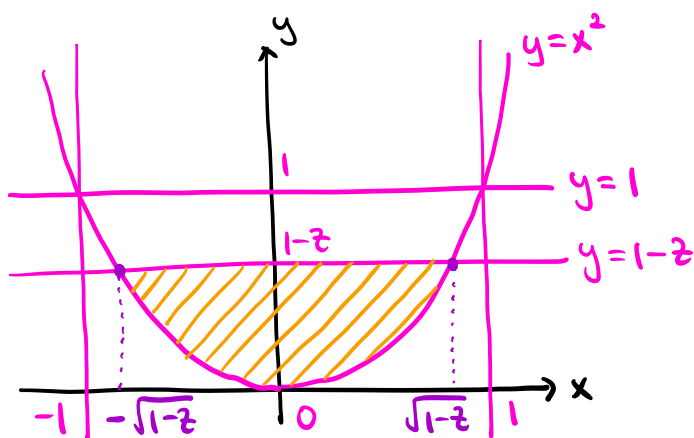The bounds are $-1 \leq x \leq 1$, $x^2 \leq y \leq 1$, $0 \leq z \leq 1-y$.

The outermost integral is with respect to dz.

$$\Rightarrow \quad 0 \leq z \leq 1-y \underset{\underset{x^2 \leq y}{\uparrow}}{\leq} 1-x^2 \underset{\underset{-1 \leq x \leq 1}{\uparrow}}{\leq} 1 \quad \Rightarrow \quad 0 \leq z \leq 1.$$

For the inner double integral, look at the cross section with <u>constant</u> z.

The boundary conditions: $-1 \leq x \leq 1$, $x^2 \leq y \leq 1$, $y \leq 1-z$.

Intersections: $y = x^2$ and $y = 1-z$

$$\Rightarrow x = \pm\sqrt{1-z}.$$

The bounds are given by

$$-\sqrt{1-z} \leq x \leq \sqrt{1-z}, \quad x^2 \leq y \leq 1-z.$$

$$\Rightarrow \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} 1 \, dz \, dy \, dx = \boxed{\int_{0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} 1 \, dy \, dx \, dz}$$